Linear and Nonlinear Realizations of Supersymmetry

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References

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• Prelude
  – Three Goldstino fields: linear/nonlinear/constrained
  – Two pion fields: linear/nonlinear
• Superspace, linear/nonlinear realization of SUSY
• Reformulate linear SUSY into nonlinear ones
• Nonlinear Goldstino field out of linear superfield
• Constrained superfields
• Low energy effective theory
• Conclusions
• Spontaneous breaking of global symmetries $\rightarrow$ massless Goldstone particles

• Properties of Goldstone particles $\leftrightarrow$ the nature of the broken and unbroken symmetries

• Strong interactions: pions $\leftrightarrow$ spontaneous breaking of chiral symmetry

• Low energy physics of pions: nonlinear realization of chiral symmetry
  – Expansion in terms of energy-momentum
• Goldstino: spontaneous breaking of global SUSY
• Supergravity: Goldstino is part of the massive gravitino
• $M_{\text{SUSY}} \ll M_P$: lower energy physics dominated by Goldstino
• Goldstino physics could be of importance at the TeV scale and tested in LHC
• Low energy physics of Goldstinos: linear SUSY/nonlinear SUSY/constrained superfields
• **Linear SUSY**, chiral fields responsible for SSB

• **Chiral super-multiplet** $\Phi \sim (\phi, \psi, F')$

\[
\begin{align*}
\delta_\xi \phi &= \sqrt{2} \xi \psi, \\
\delta_\xi \psi_\alpha &= \sqrt{2} F \xi_\alpha + i \sqrt{2} (\sigma^\mu \bar{\xi})_\alpha \partial_\mu \phi, \\
\delta_\xi F &= i \sqrt{2} \bar{\xi} \sigma^\mu \partial_\mu \psi.
\end{align*}
\]

• **The Goldstino field**
  
  – $F$-term in one $\Phi_0$ develops a nonzero VEV $\langle F_0 \rangle$, SUSY is spontaneously broken
  
  – Goldstino field $\psi_0$: zero mass and changes as

\[
\delta_\xi \psi_0_\alpha = \sqrt{2} \langle F_0 \rangle \xi_\alpha + \cdots
\]

  – If several $F$ have nonzero VEVs, realignment can be made such that only $\Phi_0$ has a nonzero VEV.
Goldstino Field in Nonlinear Realization of SUSY

- Non-chiral version

\[ \delta \xi \lambda^\alpha = \frac{1}{\kappa} \xi^\alpha - i \kappa (\lambda \sigma^\mu \bar{\xi} - \xi \sigma^\mu \bar{\lambda}) \partial_\mu \lambda^\alpha \]

\[ \delta \xi \bar{\lambda}_{\dot{\alpha}} = \frac{1}{\kappa} \bar{\xi}_{\dot{\alpha}} - i \kappa (\lambda \sigma^\mu \bar{\xi} - \xi \sigma^\mu \bar{\lambda}) \partial_\mu \bar{\lambda}_{\dot{\alpha}} \]

- Chiral version

\[ \delta \xi \bar{\lambda}_{\dot{\alpha}} = \frac{\xi_{\dot{\alpha}}}{\kappa} - 2i \kappa \bar{\lambda} \sigma^\mu \bar{\xi} \partial_\mu \bar{\lambda}_{\dot{\alpha}} \]

- Conversion

\[ \bar{\lambda}_{\dot{\alpha}}(x) = \lambda_\alpha(z), \quad z = x - i \kappa^2 \lambda(z) \sigma \bar{\lambda}(z) \]
• Conservation equation: \( \bar{D}^\dot{\alpha} J_{\alpha \dot{\alpha}} = D_\alpha X \)

• Supercurrent multiplet:

\[
J_\mu = j_\mu + \left[ \theta^\alpha \left( S_{\mu \alpha} + \frac{1}{3} (\sigma_\mu \bar{\sigma}^\rho S_\rho)_\alpha \right) + h.c. \right] + (\theta \sigma^\nu \bar{\theta}) \left( 2T_{\nu \mu} - \frac{2}{3} \eta_{\nu \mu} T - \frac{1}{4} \epsilon_{\nu \mu \rho \sigma} \partial [\rho j^\sigma] \right) \\
+ \frac{i}{2} \theta^2 \partial_\mu \bar{x} - \frac{i}{2} \bar{\theta}^2 \partial_\mu x + \ldots \\
X = x(y) + \sqrt{2} \theta \psi(y) + \theta^2 F(y)
\]

\( \partial^\mu T_{\mu \nu} = \partial^\mu S_{\mu \alpha} = 0 \), \( T_{\mu \nu} = T_{\nu \mu} \).

\( \psi_\alpha = \frac{\sqrt{2}}{3} \sigma^\mu_{\alpha \dot{\alpha}} \bar{S}_\mu^{\dot{\alpha}}, \ F = \frac{2}{3} T + i \partial_\mu j^\mu, \ y^\mu = x^\mu + i \theta \sigma^\mu \bar{\theta} \)
Work of Komargodski/Seiberg-2

- SUSY spontaneously broken, the low-energy supercurrent in terms of the massless Goldstino $G_\alpha$

\[ S_{\mu\alpha} = \sqrt{2} f \sigma_{\mu\alpha\dot{\alpha}} \bar{G}^{\dot{\alpha}} + f'(\sigma_{\mu\nu})^\beta_\alpha \partial^\nu G_\beta + \cdots \]

- Low-energy Goldstino *not* accompanied by a massless scalar, the simplest bosonic state $\sim$ two Goldstinos

- SUSY partners:
  - one Goldstino $Q^\dagger_\dot{\alpha} |0\rangle \sim$ “two Goldstinos” $Q^\dagger_1 Q^\dagger_2 |0\rangle$
  - $\psi_\phi$ creates a one Goldstino state 
    $\phi \sim x$ creates a two Goldstino state


In combination with the SUSY algebra

\[ X_{NL} = \frac{G^2}{2F} + \sqrt{2}\theta G + \theta^2 F, \quad X_{NL}^2 = 0 \]

Integrate out heavy fields by constraints

Coupling with superfields: spurion \( Y \to \frac{m_{soft}}{f} X_{NL} \)

\[
\mathcal{L}_{soft} = - \int d^4 \theta \left| \frac{X_{NL}}{f} \right|^2 (m^2)^j_i (Qe^V \bar{Q})^i_j \\
+ \int d^2 \theta \frac{X_{NL}}{f} \left( -\frac{1}{2} B_{ij} Q^i Q^j + \frac{1}{6} A_{ijk} Q^i Q^j Q^k \right) + c.c.
\]

\[
\mathcal{L}_{soft} = \int d^4 \theta \left| \frac{X_{NL}}{f} \right|^2 \xi D^\alpha W_\alpha + \int d^2 \theta \frac{X_{NL}}{f} m_\lambda W_\alpha W^\alpha + c.c.
\]
The Web of Relations

- **Relation between \( \tilde{\lambda} \) and \( \psi_0 \)**
  - \( \tilde{\lambda} \) is closely related, but not identical, to \( \psi_0 \)
  - \( \tilde{\lambda} \simeq \psi_0 \) to the leading order, if \( \kappa^{-1} = \sqrt{2}\langle F_0 \rangle \)

- **Construct \( \tilde{\lambda} \) out of \( (\phi_0, \psi_0, F_0) \subset \Phi_0 \)**

\[
\tilde{\lambda} = \frac{\psi_0}{\sqrt{2\kappa F_0}} - i \frac{\sigma^\mu \tilde{\lambda}}{F_0} \left( \partial_\mu \phi_0 - \sqrt{2}\kappa \tilde{\lambda} \partial_\mu \psi_0 + \kappa^2 \tilde{\lambda}^2 \partial_\mu F_0 \right)
\]

- **Expressing \( X_{NL} \) in the language of nonlinear SUSY**
- **Comparison with the nonlinear/linear \( \sigma \)-model**
  - \( \lambda \sim \vec{\pi} \)
  - \( (\phi_0, \psi_0, F_0) \sim \phi_a \)
  - \( X_{NL}^2 = 0 \sim \sum_a \phi_a^2 = \text{constant} \)
- **Linear $\sigma$-model**
  - SO(4)-invariant Lagrangian of the linear $\sigma$-model

\[
L = -\frac{1}{2} \partial_\mu \phi_n \partial^\mu \phi_n - \frac{M^2}{2} \phi_n \phi_n - \frac{\lambda}{4} (\phi_n \phi_n)^2
\]

\[
= -\frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2} \sigma^2 \sum_{n=1}^{4} \partial^\mu R_{n4} \partial_\mu R_{n4} - \frac{1}{2} M^2 \sigma^2 - \frac{\lambda}{4} \sigma^4
\]

\[
\phi_n(x) = R_{n4}(x)\sigma(x), \quad R^T R = 1, \quad \sigma = \sqrt{\sum_n \phi_n^2}
\]

- if $M^2 < 0$, $\sigma$ has a non-zero VEV
Nonlinear $\sigma$-model

- Redefine the fields

$$\zeta_a \equiv \frac{\phi_a}{\phi_4 + \sigma}, \quad \phi_a/\sigma = R_{a4} = \frac{2\zeta_a}{1 + \zeta^2}, \quad \phi_4/\sigma = R_{44} = \frac{1 - \zeta^2}{1 + \zeta^2}$$

$$R_{a4} = \frac{2\zeta_a}{1 + \zeta^2} = -R_{4a}, \quad R_{44} = \frac{1 - \zeta^2}{1 + \zeta^2}, \quad R_{ab} = \delta_{ab} - \frac{2\zeta_a\zeta_b}{1 + \zeta^2}$$

- Nonlinear $\sigma$-Lagrangian

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - 2\sigma^2 \vec{D}_\mu \vec{D}^\mu - \frac{1}{2} M^2 \sigma^2 - \frac{\lambda}{4} \sigma^4$$

$$\vec{D}_\mu \equiv \frac{\partial_\mu \zeta}{1 + \zeta^2}$$
- **Transformation rules in nonlinear $\sigma$-model**
  - Isospin: $\delta \vec{\phi} = \vec{\theta} \times \vec{\phi}, \delta \phi_4 = 0$
    \[ \delta \sigma = 0, \quad \delta \vec{\zeta} = \vec{\theta} \times \vec{\zeta}, \quad \delta \vec{D}_\mu = \vec{\theta} \times \vec{D}_\mu \]
  - Axial isospin: $\delta \vec{\phi} = 2\vec{\epsilon}\phi_4, \quad \delta \phi_4 = -2\vec{\epsilon} \cdot \vec{\phi}$
    \[ \delta \sigma = 0, \quad \delta \vec{\zeta} = \epsilon(1 - \vec{\zeta}^2) + 2\vec{\zeta}(\vec{\epsilon} \cdot \vec{\zeta}), \quad \delta \vec{D}_\mu = 2(\vec{\zeta} \times \vec{\epsilon}) \times \vec{D}_\mu \]

- **Take** $F = 2\langle \sigma \rangle$ and $\vec{\pi} \equiv F\vec{\zeta}$,
  (Set $\sigma$ to its VEV, constraint: $\sum_a \phi_a^2 = \langle \sigma \rangle^2$)
  \[ \mathcal{L} = -\frac{F^2}{2} \vec{D}_\mu \vec{D}^\mu = -\frac{1}{2} \frac{\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi}}{1 + \vec{\pi}^2/F^2} \]
Superspace, Translations, Induced Realization

- Superalgebra
  \[
  \{ \hat{Q}_\alpha, \bar{Q}_{\dot{\alpha}} \} = -2i\sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu, \quad \{ \hat{Q}_\alpha, \bar{Q}_\beta \} = 0, \quad \{ \bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}} \} = 0
  \]

- Supergroup element in superspace \((x, \theta, \bar{\theta})\)
  \[
  G(x, \theta, \bar{\theta}) = \exp \left[ i(-x^\mu P_\mu + \theta \hat{Q} + \bar{\theta} \bar{Q}) \right]
  \]

- Multiplication
  \[
  G(0, \xi, \bar{\xi})G(x, \theta, \bar{\theta}) = G(x + i(\theta \sigma \bar{\xi} - \xi \sigma \bar{\theta}), \theta + \xi, \bar{\theta} + \bar{\xi})
  \]

- Translation in superspace
  \[
  x' = x + i(\theta \sigma \bar{\xi} - \xi \sigma \bar{\theta}), \quad \theta' = \theta + \xi, \quad \bar{\theta}' = \bar{\theta} + \bar{\xi}
  \]

  generated by \(\xi Q + \bar{\xi} \bar{Q}\)

  \[
  Q_\alpha = \partial_\alpha - i(\sigma^\mu \bar{\theta})_\alpha \partial_\mu, \quad \bar{Q}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} + i(\theta \sigma^\mu)_{\dot{\alpha}} \partial_\mu
  \]
Linear Realization of SUSY

- Induced linear realization of the superalgebra
  \[ \{ Q_\alpha, \bar{Q}_{\dot{\alpha}} \} = 2i\sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu, \quad \{ Q_\alpha, Q_\beta \} = 0, \quad \{ \bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}} \} = 0 \]

- Change in sign, the order of multiplication reversed

- Superfield:
  \[
  F(x, \theta, \bar{\theta}) = f(x) + \theta \phi(x) + \bar{\theta} \bar{\chi}(x) + \theta \theta m(x) + \bar{\theta} \bar{\theta} n(x) \\
  + \theta \sigma^\mu \bar{\theta} v_\mu(x) + \theta \theta \bar{\theta} \bar{\lambda}(x) + \bar{\theta} \bar{\theta} \psi(x) + \theta \theta \bar{\theta} \bar{\theta} d(x)
  \]

- Linear transformation mixes different components
  \[
  \delta_\xi F(x, \theta, \bar{\theta}) = (\xi Q + \bar{\xi} \bar{Q}) F(x, \theta, \bar{\theta}) \\
  = \delta_\xi f(x) + \theta \delta_\xi \phi(x) + \bar{\theta} \delta_\xi \bar{\chi}(x) + \theta \theta \delta_\xi m(x) + \bar{\theta} \bar{\theta} \delta_\xi n(x) \\
  + \theta \sigma^\mu \bar{\theta} \delta_\xi v_\mu(x) + \theta \theta \bar{\theta} \delta_\xi \bar{\lambda}(x) + \bar{\theta} \bar{\theta} \theta \delta_\xi \psi(x) + \theta \theta \bar{\theta} \bar{\theta} \delta_\xi d(x)
  \]
Nonlinear Realizations of SUSY-1

• Induced nonlinear realization: \( \theta \rightarrow \kappa \lambda(x) \)

\[
\lambda'(x') = \lambda(x) + \frac{1}{\kappa} \xi, \quad \bar{\lambda}'(x') = \bar{\lambda}(x) + \frac{1}{\kappa} \bar{\xi}
\]

• Infinitesimal changes

\[
\left[ \mathcal{V}_\xi^\mu(x) = \kappa (\lambda \sigma^\mu \bar{\xi} - \xi \sigma^\mu \bar{\lambda}) \right]
\]

\[
\delta_\xi \lambda^\alpha = \frac{1}{\kappa} \xi^\alpha - i \mathcal{V}_\xi^\mu(x) \partial_\mu \lambda^\alpha, \quad \delta_\xi \bar{\lambda}_{\dot{\alpha}} = \frac{1}{\kappa} \bar{\xi}_{\dot{\alpha}} - i \mathcal{V}_\xi^\mu(x) \partial_\mu \bar{\lambda}_{\dot{\alpha}}
\]

• The SUSY algebra is closed

\[
(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta) \lambda^\alpha = -2i (\eta \sigma^\mu \bar{\xi} - \xi \sigma^\mu \bar{\eta}) \partial_\mu \lambda^\alpha
\]

• Matter fields

\[
\delta_\xi f(x) = -i \mathcal{V}_\xi^\mu(x) \partial_\mu f(x)
\]

\[
(\delta_\eta \delta_\xi - \delta_\xi \delta_\eta) f = -2i (\eta \sigma^\mu \bar{\xi} - \xi \sigma^\mu \bar{\eta}) \partial_\mu f
\]
Nonlinear Realizations of SUSY-2

- Taking \((x', \theta', \bar{\theta}')\) as functions of \((x, \theta, \bar{\theta})\)
  
  \[
  dx'^\mu = dx^\mu + id\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu d\bar{\theta}
  \]

  \[
  d\theta'^\alpha = d\theta^\alpha, \quad d\bar{\theta}'_{\dot{\alpha}} = d\bar{\theta}_{\dot{\alpha}}
  \]

- Define differentials
  
  \[
  e^\mu = dx^\mu - id\theta\sigma^\mu\bar{\theta} + i\theta\sigma^\mu d\bar{\theta}
  \]

  \[
  e^\alpha = d\theta^\alpha, \quad e_{\dot{\alpha}} = d\bar{\theta}_{\dot{\alpha}}
  \]

- \((x, \theta, \bar{\theta}) \rightarrow (x', \theta', \bar{\theta}'), \quad e^\mu \rightarrow e'^\mu\)
  
  \[
  e'^\mu = dx'^\mu - id\theta'\sigma^\mu\bar{\theta}' + i\theta'\sigma^\mu d\bar{\theta}' = e^\mu
  \]

  It is invariant.
Nonlinear Realizations of SUSY-3

- **Akulov-Volkov Lagrangian**
  - Substituting \( \theta = \kappa \lambda \) and \( d\theta = \kappa (\partial \lambda / \partial x^\mu) dx^\mu \)
  
  \[
e^\mu \rightarrow dx^\nu \left[ \delta^\mu_\nu - i\kappa^2 \partial_\nu \lambda \sigma^\mu \bar{\lambda} + i\kappa^2 \lambda \sigma^\mu \partial_\nu \bar{\lambda} \right] = dx^\nu T^\mu_\nu
  \]
  - A-V Lagrangian
    
    \[\mathcal{L} = -\frac{1}{\kappa^2} \text{det } T\]
  - \( \mathcal{L} \) changes by a total derivative
    
    \[\delta_\xi \text{det } T = -i\kappa \partial_\mu \left[ (\lambda \sigma^\mu \bar{\xi} - \bar{\xi} \sigma^\mu \lambda) \text{det } T \right]\]

- **Any SUSY non-invariant theory can be prompted to an (nonlinearly) invariant one**
  (Low energy effective theory, later)
From Linear to Nonlinear SUSY-1

- **General linear super-multiplet**
  \[
  \Phi^\sigma_k(x, \theta, \bar{\theta}) = e^{-\kappa \lambda(x) Q - \kappa \bar{\lambda}(x) \bar{Q}} \Phi_k(x, \theta, \bar{\theta}) = \Phi_k(\tilde{x}, \tilde{\theta}, \bar{\tilde{\theta}})
  \]
  \[
  \tilde{x} = x + i\kappa \lambda(x) \sigma \bar{\theta} - i\kappa \theta \sigma \bar{\lambda}(x)
  \]
  \[
  \tilde{\theta} = \theta - \kappa \lambda(x), \quad \bar{\tilde{\theta}} = \bar{\theta} - \kappa \bar{\lambda}(x)
  \]

- **All components of** \(\Phi^\sigma_k\) **change as matter fields**
  \[
  \delta_\xi \Phi^\sigma_k(x, \theta, \bar{\theta}) = -iv^\nu_\xi(x) \frac{\partial}{\partial x^\nu} \Phi^\sigma_k(x, \theta, \bar{\theta})
  \]
From Linear to Nonlinear SUSY-2

• Generic action, linear
\[ S_{gen} = \int d^4x d^4\theta \mathcal{L}_{gen} \times (\Phi_t(x, \theta), \Phi_k(x, \theta, \bar{\theta}), D_\alpha \Phi_k, D_\alpha D_\beta \Phi_k, \ldots) \]

• Generic action, nonlinear
\[ S_{gen} = \int d^4x d^4\theta Ber(x, \theta, \bar{\theta}) \times \mathcal{L}_{gen} \left( \Phi^\sigma_t(x, \theta, \bar{\theta}), \Phi^\sigma_k(x, \theta, \bar{\theta}), \triangle_\alpha \Phi^\sigma_k, \triangle_\alpha \triangle_\beta \Phi^\sigma_k, \ldots \right) \]

\[ Ber(x, \theta, \bar{\theta}) = \text{det } T(x) \text{det } M(x, \theta, \bar{\theta}) \]

\[ M^\nu_\mu(x, \theta, \bar{\theta}) = \delta^\nu_\mu + i\kappa \nabla_\mu \lambda(x) \sigma^\nu \bar{\theta} - i\kappa \theta \sigma^\nu \nabla_\mu \bar{\lambda}(x) \]

• Covar. derivatives:
\[ \nabla^\nu = (T^{-1})^\nu_\rho \partial_\rho, \ \triangle_\alpha = \frac{\partial}{\partial \theta^\alpha} + i(\sigma_\mu \bar{\theta})_\alpha \triangle^\mu \]
\[ \triangle^\mu = (M^{-1})^\mu_\nu \left( \nabla^\nu + \nabla^\nu \lambda(x) \frac{\partial}{\partial \theta} + \nabla^\nu \bar{\lambda}(x) \frac{\partial}{\partial \bar{\theta}} \right) \]
• Chiral superfield $\Phi_t$

$$\Phi_t(x, \theta, \bar{\theta}) = \exp \left( i \theta \sigma^\mu \bar{\theta} \partial_{\mu} \right) S_t(x, \theta)$$

$$\Phi^\sigma_t(x, \theta, \bar{\theta}) = L^+ \left( \partial/\partial x, \partial/\partial \theta \right) S^\sigma_t(x, \theta)$$

$$S^\sigma_t(x, \theta) = S_t \left( \tilde{x}^+, \tilde{\theta} \right)$$

$$\tilde{x}^+ = x - 2i \kappa \theta \sigma \bar{\lambda}(x) + i \kappa^2 \lambda(x) \sigma \bar{\lambda}(x)$$

$$L^+ \left( \partial/\partial x, \partial/\partial \theta \right) = 1 + i \theta \sigma^\mu \bar{\theta} \triangle^+_{\mu} + \frac{1}{4} \theta^2 \bar{\theta}^2 \triangle^+_{\mu} \triangle^+_{\mu} + \frac{1}{4} \theta^2 \bar{\theta}^2 \triangle^+_{\mu} \triangle^+_{\mu}$$

$$M^{+\nu}_{\mu}(x, \theta) = \delta^\nu_{\mu} - 2i \kappa \theta \sigma_{\mu} \nabla^\nu \bar{\lambda}, \quad \triangle^+_{\alpha} = \frac{\partial}{\partial \theta^\alpha} + i (\bar{\theta} \sigma^\mu)_{\alpha} \triangle^+_{\mu},$$

$$\triangle^+_{\mu} = (M^{-1}_+)^\mu_{\nu} \left( \nabla^\nu + \nabla^\nu \lambda(x) \frac{\partial}{\partial \theta} \right)$$
From Linear to Nonlinear SUSY-4

- Chiral part of the action, linear
  \[ S_{ch} = \int d^4x \left( d^2\theta \mathcal{L}_{ch}(S_t(x, \theta)) + C.C \right) \]

- Chiral part of the action, nonlinear
  \[ S_{ch} = \int d^4x \left( d^2\theta \text{Ber}^+ (x, \theta) \mathcal{L}_{ch}(S^\sigma_t(x, \theta)) + C.C \right) \]
  \[ \text{Ber}^+ (x, \theta) = \det T(x) \det M^+(x, \theta) \]
From Linear to Nonlinear SUSY: Recapture

- Linear superfields to nonlinear ones
  \[ \Omega^\sigma = \exp\left[ -\kappa (\lambda Q + \bar{\lambda} \bar{Q}) \right] \Omega \]

- SUSY transformation rules for \( \Omega^\sigma \)
  \[ \delta_\xi \Omega^\sigma = -i(\lambda^\sigma \mu \bar{\xi} - \xi^\sigma \mu \bar{\lambda}) \partial_\mu \Omega^\sigma \]

- All component fields in \( \Omega^\sigma \) transform into themselves
- Any of them can be integrated out without breaking SUSY, via e.o.m. (tree level) or matching (QM), producing high dimensional operators
- Extremely heavy ones: set to zero directly
- Whether and how to integrate out a field are dynamical questions
Construct the Nonlinear Goldstino Field $\lambda$

- Generic OR model: $(\Phi_0 \rightarrow S_0$ by ridding of $i\theta \sigma \bar{\theta})$
  \[ S_0(x, \theta) = \phi_0(x) + \sqrt{2}\theta \psi_0(x) + \theta^2 F_0(x) \]

- Essential: $\langle F_0 \rangle \neq 0$

- The corresponding nonlinear super-multiplet
  \[ S_0^\sigma = S_0(x - 2i\kappa \theta \sigma \bar{\lambda}(x) + i\kappa^2 \lambda(x)\sigma \bar{\lambda}(x), \theta - \kappa \lambda(x)) \]

- Construct $\lambda$ out of the components of $S_0$: demanding $\psi_0^\sigma$ to vanish and re-express $\lambda$ in terms of $\tilde{\lambda}$
  \[ \tilde{\lambda} = \frac{\psi_0}{\sqrt{2}\kappa F_0} - i\frac{\sigma^\mu \bar{\tilde{\lambda}}}{F_0} (\partial_\mu \phi_0 - \sqrt{2}\kappa \tilde{\lambda} \partial_\mu \psi_0 + \kappa^2 \tilde{\lambda}^2 \partial_\mu F_0) \]

- The analog of representing $\vec{\pi}$ in terms of $\phi_a$ in $\sigma$-models
Comments on the Construction

- Taking $\kappa^{-1} = \sqrt{2} \langle F_0 \rangle$: $\tilde{\lambda} \simeq \psi_0$ to the leading order
- $\tilde{\lambda} (\lambda)$ transforms properly
- $\psi_0^\sigma = 0$ in $\Phi_0^\sigma = \exp \left[ -\kappa \left( \lambda Q + \bar{\lambda} \bar{Q} \right) \right] \Phi_0$ when this $\lambda$ is used, it is realized by the definition of $\lambda$
- $\psi_0$ cannot be dropped by the reasoning of dynamics for it's not heavy, it is actually massless
- Feasibility due to the SUSY algebras
- Can always construct a $\lambda$ for any chiral super-multiplet, but cannot be used to separate the Goldstino field from the others
• Standard procedure, with *this* definition of $\lambda$

• No explicit form of $\lambda$ is needed, the key element is $\psi^\sigma_0 = 0$, which is all needed

• In the process, the Goldstino field disappears from the original Lagrangian, but reemerges in the Jacobian of the transformation and covariant derivatives

• Vertices with Goldstino fields carry at least one space-time derivative, as one would have expected

• All fields are kept, heavy ones can be integrated out, via e.o.m. or matching
Mass Spectrum in Nonlinear Lagrangians

- Space-time derivatives are not allowed in potential terms, Goldstino field is absent in the nonlinear version
- Potential terms in the nonlinear version
  \[ \int d^4x (d^2 \theta W(S_t^s, S_0^s) + h.c.) \]
- The same structure as the linear version
  \[ \int d^4x (d^2 \theta W(S_t, S_0) + h.c.) \]
  \( \psi_0 \) is massless: no bilinear terms \( \psi_0 \psi_0 \) or \( \psi_0 \psi_i \)
- The mass spectrum is not changed by going from the linear version to nonlinear one by setting \( \psi_0^s = 0 \)
Construction of the Goldstino field in F-I models

- Abelian gauge field
  \[ V = D\theta^2\bar{\theta}^2 + \chi\theta\bar{\theta}^2 + \bar{\chi}\bar{\theta}\theta^2 + \cdots \]
- Non-zero VEV for \( D \rightarrow \) SUSY spontaneously broken
- \( \chi \): massless, the Goldstino field
- Define a nonlinear Goldstino field \( \lambda \) by demanding \( \chi^\sigma = 0 \) in nonlinearly realized super-multiplet \( V^\sigma \)
- Problems about gauge and supergravity
Constrained Field for the Goldstino-1

- Goldstino field in a linearly chiral superfield $X_{NL}$
- $X_{NL}^2 = 0$, to rid of the scalar component
  - Supersymmetry structure and its breaking
  - $X_{NL} = \frac{G^2}{2F} + \sqrt{2}\theta G + \theta^2 F$
- Define $\lambda^{NL} = G/\sqrt{2}\kappa F$
  \[
  \delta_\xi \lambda^{NL}_\alpha = \frac{\xi_\alpha}{\kappa} - 2i\kappa\lambda^{NL}_\alpha \sigma^\mu \bar{\xi} \partial_\mu \lambda^{NL}_\alpha
  \]
- $\lambda^{NL}$ transforms in exactly the same way as $\tilde{\lambda}$
- $\lambda^{NL} = \tilde{\lambda}$ and $X_{NL} = F\Theta^2$, $\Theta = \theta + \kappa\tilde{\lambda}$
Constrained Field for the Goldstino-2

- Self consistent check:
  - $X_{NL}^\sigma = \theta^2 F^\sigma$
  - $\lambda$ disappears in the nonlinearly realized super-multiplet

- Reverse the logic
  - For any chiral superfield $\Phi = \phi + \sqrt{2}\theta\psi + \theta^2 F$
    define $\lambda^\Phi = \psi/\sqrt{2}\kappa F$
  
  $\delta_\xi \lambda^\alpha = \frac{\xi^\alpha}{\kappa} - 2i\kappa \lambda^\Phi \sigma^\mu \bar{\xi} \partial_\mu \lambda^\alpha + \frac{i}{\kappa F} (\sigma^\mu \bar{\xi})^\alpha_\alpha \partial_\mu \left( \phi - \frac{\psi^2}{2F} \right)$

  - Demanding $\lambda^\Phi$ to transform in the same way as that of $\bar{\lambda}$, one obtains $\phi = \psi^2/2F$ and $\Phi^2 = 0$
Constrained Field for the Goldstino-3

• Prompt \( \lambda \) to a linear superfield

\[
\Lambda(\lambda) = \exp(\theta Q + \bar{\theta}\bar{Q}) \times \lambda
\]

• Construct two chiral fields out of \( \Lambda \) and \( \bar{\Lambda} \)

\[
\Phi_2 = -\frac{1}{4}\bar{D}^2\Lambda\Lambda, \quad \Phi_4 = -\frac{1}{4}\bar{D}^2\Lambda\Lambda\bar{\Lambda}\bar{\Lambda}.
\]

\[
\Phi_2 = f_2(\lambda)\Theta^2, \quad \Phi_4 = f_4(\lambda)\Theta^2,
\]

• \( f_2, f_4 \): definite functions of \( \lambda \)

• \( f_4 \): the AV Lagrangian up to an overall constant and possible total derivative terms

• \( f_4/f_2 \) and \( F/f_2 \) transform as matter fields
Constrained Field for the Goldstino-4

- Some history:
  \[ \Phi_4 \bar{D}^2 \Phi_4 \sim \Phi_4 \]
  while \( \Phi_2 \) does not have such relation
- The rationale to choose \( \Phi_4 \) instead of \( \Phi_2 \) to be the superfield for Goldstino
- They differ only by a matter field in the standard realization
- Obvious in retrospect, since \( \Phi_2^2 = \Phi_4^2 = 0 \), the same form of factorization
Constrained Field for the Goldstino-4

- Real superfield $V_4 = \Lambda^2 \bar{\Lambda}^2$
- $V_4^2 = 0$ and $V_4 = f_4(\lambda) \Theta^2 \bar{\Theta}^2$, $\Theta = \theta + \kappa \lambda$
- $f_4$: the AV Lagrangian up to an overall constant and possible total derivative terms
- $V^2 = 0$ cannot be preserved under a general gauge transformation
- For any $V = D\theta^2 \bar{\theta}^2 + \chi \theta \bar{\theta}^2 + \bar{\chi} \bar{\theta} \theta^2 + \cdots$
  - Define $\lambda^V = \chi/2\kappa D$
  - $\delta_\xi \lambda^V = \frac{\xi_\alpha}{\kappa} - i(\lambda^V \sigma^\mu \bar{\xi} - \xi \sigma^\mu \bar{\lambda}^V) \partial_\mu \lambda^V + \frac{i}{D}$ (total derivatives)
  - Demanding $\lambda^V$ to transform in the same way as that of $\lambda$, one gets $V = D\Theta^2 \bar{\Theta}^2$
  - $D/f_4$ transforms as a matter field
The constraint to rid of the scalar component in a chiral superfield, $Q_{NL} = \phi_q + \sqrt{2}\theta\psi_q + \theta^2 F_q$

$$X_{NL}Q_{NL} = 0$$

From which, one gets

$$\phi_q = \frac{\psi_q G}{F} - \frac{G^2}{2F^2}F_q$$

Equivalent to the constraint $\phi^\sigma = 0$
Low Energy Effective Theory-1

- Prompt $\lambda$ and matter fields to linear super-multiplets
  \[ \Lambda(\lambda) = \exp(\theta Q + \bar{\theta} \bar{Q}) \times \lambda \]
  \[ \Omega(\lambda) = \exp(\theta Q + \bar{\theta} \bar{Q}) \times \varphi \]
- SUSY non-invariant action $\rightarrow$ SUSY invariant one
  \[ \int d^4x L(\partial_\mu \varphi, \varphi) \]
  \[ \downarrow \]
  \[ \kappa^4 \int d^4x d^4\theta \Lambda \Lambda \bar{\Lambda} \bar{\Lambda} L(\partial_\mu \Omega, \Omega) \]
- $\Lambda(x) = \kappa^{-1} \theta' = \kappa^{-1} \theta + \lambda(z)$
  $\Omega(x) = \varphi(z), \ z = x - i\kappa \lambda(z) \sigma \bar{\theta} + i\kappa \theta \sigma \bar{\lambda}(z)$
Integrate out the Grassmann variables: $(x, \theta) \rightarrow (z, \theta')$

$$S = \int d^4x (\text{det } T) \ L(\nabla_\mu \phi, \phi)$$

- Same results by changing $\partial_\mu \rightarrow \nabla_\mu$ and inserting $\text{det } T$
- $S$ is invariant under nonlinear SUSY transformations
- Integrations over the Grassmann variables can always be carried out in a similar manner for arbitrary functionals of $\Lambda$ and $\Omega \rightarrow$ extra operators for effective theories
• Subtleties for gauge theories
• Wess-Zumino gauge, starting with the transformation
  \[ \delta \xi A_\mu = -i\chi \sigma_\mu \bar{\xi} + i\xi \sigma_\mu \bar{\chi} \]
  \[ \delta \xi \chi_\alpha = \sigma^{\mu\nu} \xi_\alpha F_{\mu\nu} + i\xi D \]
  \[ \delta \xi D = -D_\mu \chi \sigma_\mu \bar{\xi} - \xi \sigma_\mu D_\mu \bar{\chi} \]
• \( D_\mu = \partial_\mu - iA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - i\partial_\nu A_\mu - i[A_\mu, A_\nu] \)
• Construct four superfields \( V_\mu = \exp(\theta Q + \bar{\theta} \bar{Q}) \times A_\mu \)
• Four nonlinearly realized superfields
  \[ \tilde{V}_\mu = \exp \left[ -\kappa(\lambda Q + \bar{\lambda} \bar{Q}) \right] \times V_\mu = \tilde{A}_\mu + i\theta \sigma_\mu \tilde{\chi} - i\bar{\chi} \sigma_\mu \bar{\theta} + \cdots \]
• Transformation rules of $\tilde{A}_\mu$

$$\delta_\xi \tilde{A}_\mu = -i\kappa v^\nu_\xi \tilde{F}_{\nu\mu}$$

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - i\partial_\nu \tilde{A}_\mu - i[\tilde{A}_\mu, \tilde{A}_\nu]$$

• This can be rewritten as

$$\delta_\xi \tilde{A}_\mu = -i\kappa v^\nu_\xi \partial_\nu \tilde{A}_\mu - i\kappa \partial_\mu v^\nu_\xi \tilde{A}_\nu + D_\mu (i\kappa v^\nu_\xi \tilde{A}_\nu)$$

• The last term can be compensated by a gauge transformation of the parameter $-i\kappa v^\nu_\xi \tilde{A}_\nu$

• Under this combination of SUSY and gauge transformations

$$\delta'_\xi \tilde{A}_\mu = -i\kappa v^\nu_\xi \partial_\nu \tilde{A}_\mu - i\kappa \partial_\mu v^\nu_\xi \tilde{A}_\nu$$
• Define

\[ D_\mu = (T^{-1})^\nu_\mu D_\nu = (T^{-1})^\nu_\mu (\partial_\nu - iA_\nu) \]

\[ F_{\mu\nu} = (T^{-1})^\rho_\mu (T^{-1})^\sigma_\nu (\partial_\rho A_\sigma - \partial_\sigma A_\rho - i[A_\rho, A_\sigma]) \]

• \( D_\mu \) and \( F_{\mu\nu} \) transform covariantly under both SUSY and gauge rotation

• Substitute \( D_\mu \rightarrow D_\mu \) and \( F_{\mu\nu} \rightarrow F_{\mu\nu} \):
  non-SUSY Lagrangians \( \rightarrow \) SUSY Lagrangians
  (with gauge invariance)
Conclusions

• Construct $\tilde{\lambda}$ out of $(\phi_0, \psi_0, F_0) \subset \Phi_0$

$$\tilde{\lambda} = \frac{\psi_0}{\sqrt{2\kappa F_0}} - i \frac{\sigma^\mu \tilde{\lambda}}{F_0} (\partial_\mu \phi_0 - \sqrt{2\kappa} \tilde{\lambda} \partial_\mu \psi_0 + \kappa^2 \tilde{\lambda}^2 \partial_\mu F_0)$$

• Linear SUSY theories reformulated into non-linear ones
• Goldstino field disappears in the process, reemerges in the Jacobian and covariant derivatives
• Vertices with Goldstinos carry space-time derivatives
• Heavy ones can be integrated out, via e.o.m. or matching, without breaking SUSY
• Constrained superfield reformulated in terms of the standard realization: $X^2_{NL} = 0 \rightarrow \tilde{\lambda} = \psi/\sqrt{2\kappa F}$
• SUSY non-invariant theories can be prompted to non-linearly invariant ones
Thank You!