Teleparallel dark energy

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JA Gu, CC Lee, CQ Geng arXiv:1204.4048
Outline

- Teleparallel Gravity
- Teleparallel Dark Energy model
- Observational Constraints
- Tracker Behavior
Teleparallel Gravity

What is the feature of teleparallel gravity?

- An alternative theory of gravity, which is equivalent to General Relativity.
- This is a curvatureless gravity theory, and the gravitational effect comes from torsion instead of curvature.
The dynamical variable of teleparallel gravity is the vierbein fields $e_A(x^\mu)$, which form an orthonormal basis for the tangent space at each point $x^\mu$ of the manifold: $e_A \cdot e_B = \eta_{AB}$, where $\eta_{AB} = diag(1, -1, -1, -1)$.

Notation:
Greek indices $\mu, \nu, ...$ : coordinate space-time.
Latin indices $A, B, ...$ : tangent space-time.

The relationship between metric and vierbein fields is

$$g_{\mu\nu}(x) = \eta_{AB} e^A_{\mu}(x) e^B_{\nu}(x).$$
Teleparallel Gravity
A brief introduction

- Weitzenböck connection: a curvatureless connection

\[ w^\lambda \]
\[ \Gamma_{\nu\mu} \equiv e_A^\lambda \partial_\mu e^A_\nu \]

- The torsion tensor is defined as

\[ T^\lambda_{\mu\nu} \equiv w^\lambda_{\nu\mu} - w^\lambda_{\mu\nu} = e_A^\lambda (\partial_\mu e^A_\nu - \partial_\nu e^A_\mu). \]

- Under Weitzenböck connection, the Riemann tensor vanishes:

\[ R^\rho_{\mu\sigma\nu} = w^\rho_{\mu\nu,\sigma} - w^\rho_{\mu\sigma,\nu} + w^\rho_{\delta\sigma} w^\delta_{\mu\nu} - w^\rho_{\delta\nu} w^\delta_{\mu\sigma} = 0. \]
We can construct the “teleparallel Lagrangian” by using the torsion tensor,
\[ \mathcal{L}_T = T = a_1 T^{\rho\mu\nu} T_{\rho\mu\nu} + a_2 T^{\rho\mu\nu} T_{\nu\mu\rho} + a_3 T^\rho_{\rho\mu} T^\mu_{\nu\nu}. \]

It is a good approach of General Relativity when we choose the suitable parameters \( a_1 = \frac{1}{4}, a_2 = \frac{1}{2} \) and \( a_3 = -1 \):

\[ \tilde{R} = -T - 2\nabla^\mu T^\nu_{\mu\nu}, \]

where \( \tilde{R} \) is constructed by Levi-Civita connection.
The action of teleparallel gravity is

\[ S = \int d^4x e \left[ \frac{T}{2\kappa^2} + \mathcal{L}_M \right], \]

where \( e = \text{det} (e^A_\mu) = \sqrt{-g} \).

Varying this action respect to the vierbein fields gives the field equation

\[ e^{-1} \partial_\mu (ee^\rho_A S^\rho_{\mu\nu}) - e^\lambda_A T^\rho_\mu \lambda S^\nu_\mu - \frac{1}{4} e^\nu_A T = \frac{\kappa^2}{2} e^\rho_A \mathbf{T}^\rho_\nu, \]

where \( \mathbf{T}^\rho_\nu \) stands for the energy-momentum tensor and

\[ S^\rho_{\mu\nu} = \frac{1}{4} (T^\nu_\rho - T^\mu_\rho + T_\rho^{\mu\nu}) + \frac{1}{2} \left( \delta^\mu_\rho T^{\alpha\nu}_\alpha - \delta^\nu_\rho T^{\alpha\mu}_\alpha \right). \]
Teleparallel Dark Energy
What is teleparallel dark energy model?

- Teleparallel dark energy model is a dark energy model, which can explain the late time accelerating universe.
- This model combines quintessence model with teleparallel gravity.
- This model differs from quintessence model when we turn on the non-minimal coupling term.
Quintessence is one of the most popular dark energy model.

The generalized quintessence model action is given by

$$S = \int d^4 x \sqrt{-g} \left[ \frac{R}{2\kappa^2} + \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi + \xi R \phi^2 \right) - V(\phi) + \mathcal{L}_M \right]$$

Under the flat Friedmann-Robertson-Walker (FRW) background $ds^2 = dt^2 - a^2(t)d\vec{x}^2$, the effective energy and pressure density can be defined as

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) + 6\xi H \phi \dot{\phi} + 3\xi H^2 \phi^2,$$

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) + \xi \left( 2\dot{H} + 3H^2 \right) \phi^2 + 4\xi H \phi \dot{\phi} + 2\xi \phi \ddot{\phi} + 2\xi \dot{\phi}^2$$
Teleparallel Dark Energy

- Similar to quintessence model, we can construct teleparallel dark energy model, and the action is given by
  \[
  S = \int d^4 x e \left[ \frac{T}{2\kappa^2} + \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi + \xi T \phi^2 \right) - V(\phi) + \mathcal{L}_M \right].
  \]

- Variation of action with respect to the vierbein fields yields the field equation
  \[
  \left( \frac{2}{\kappa^2} + 2\xi \phi^2 \right) \left[ e^{-1} \partial_\mu (ee_A^\rho S_\rho^{\mu\nu}) - e_A^\chi T_\mu^\rho \lambda S_\rho^{\nu\mu} - \frac{1}{4} e_A^\nu T \right] - e_A^\nu \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] + e_A^\mu \partial^\nu \phi \partial_\mu \phi
  \]
  \[
  + 4\xi e_A^\rho S_\rho^{\mu\nu} \phi (\partial_\mu \phi) = e_A^\rho \text{em} T_\rho^\nu.
  \]
Again, the effective energy and pressure density under FRW metric \((e^A_\mu = \text{diag}(1, a, a, a))\) are

\[
\begin{align*}
\rho_\phi &= \frac{1}{2} \dot{\phi}^2 + V(\phi) - 3\xi H^2 \phi^2, \\
p_\phi &= \frac{1}{2} \dot{\phi}^2 - V(\phi) + 4\xi H \phi \dot{\phi} + \xi \left(3H^2 + 2\dot{H}\right) \phi^2.
\end{align*}
\]

Variation of action with respect to the scalar field gives us the equation of motion of the scalar field

\[
\ddot{\phi} + 3H \dot{\phi} + 6\xi H^2 \phi + V'(\phi) = 0.
\]

These equations lead to the continuity equation

\[
\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = 0,
\]

where \(w_\phi\) is the equation of state of the scalar field, which is defined as \(w_\phi \equiv \frac{p_\phi}{\rho_\phi}\).
In the minimal coupling case ($\xi = 0$), the teleparallel dark energy is equivalent to quintessence model.

However, these two models are different theories when we turn on the non-minimal coupling constant ($\xi \neq 0$).

Teleparallel dark energy model can cross the phantom-divide easily.

Similar to $f(T)$ theory, this model has the local Lorentz violation problem.
We would like to test teleparallel dark energy model by using the SNIa, BAO and CMB data. These observational data can tell us whether this is a suitable model for dark energy or not.

We consider three kinds of potential cases:
- Power-Law potential: \( V(\phi) = V_0 \phi^4 \)
- Exponential potential: \( V(\phi) = V_0 e^{-\kappa\phi} \)
- Inverse hyperbolic cosine potential: \( V(\phi) = \frac{V_0}{\cosh(\kappa \phi)} \)
Potential: $V(\phi) = V_0 \phi^4$

Left: fixing $\Omega_m = 27\%$, the best fit locates at $h \simeq 0.7$, 
$\xi \simeq -0.42$, $w_\phi \simeq -0.96$ and $\chi^2 \simeq 543.9$

Right: fixing $\xi = -0.41$, the best fit locates at $h \simeq 0.7$, 
$\Omega_m \simeq 28.0\%$, $w_\phi \simeq -0.99$ and $\chi^2 \simeq 544.5$
Teleparallel Dark Energy: Observational Constraints

- Potential: $V(\phi) = V_0 e^{-\kappa \phi}$

- Left: fixing $\Omega_m = 27\%$, the best fit locates at $h \simeq 0.7$, $\xi \simeq -0.41$, $w_\phi \simeq -1.04$ and $\chi^2 \simeq 544.3$

- Right: fixing $\xi = -0.41$, the best fit locates at $h \simeq 0.7$, $\Omega_m \simeq 27.1\%$, $w_\phi \simeq -1.07$ and $\chi^2 \simeq 544.6$
Potential: \( V(\phi) = \frac{V_0}{cosh(\kappa \phi)} \)

Left: fixing \( \Omega_m = 27\% \), the best fit locates at \( h \simeq 0.7 \), \( \xi \simeq -0.38 \), \( w_\phi \simeq -1.05 \) and \( \chi^2 \simeq 544.8 \)

Right: fixing \( \xi = -0.41 \), the best fit locates at \( h \simeq 0.7 \), \( \Omega_m \simeq 26.7\% \), \( w_\phi \simeq -1.03 \) and \( \chi^2 \simeq 545.1 \)
Summary

- Teleparallel gravity is an alternative gravity theory of the universe.
- Teleparallel dark energy model is equivalent to quintessence model happens at the minimal coupling case ($\xi = 0$), but it has a different behavior when we include a non-minimal coupling term ($\xi \neq 0$).
- We show that the equation of state of teleparallel dark energy model can cross the phantom-divide easily.
- The observational constraints show a good result on this model. This model is suitable for the late-time accelerating universe.
Tracker Behavior
Basic Idea and Features

- Potential-free: $V(\phi) = 0$.
- Analytic solutions in the radiation (RD), matter (MD), and scalar field (SD) dominated eras.
- The tracker behavior for $w_\phi$ in the RD and MD eras.
The field equation of gravity and scalar field lead to

$$\ddot{\phi} + 3H \dot{\phi} + 6\xi H^2 \phi = 0,$$

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3} (\rho_\phi + \rho_m + \rho_r),$$

$$\dot{H} = -\frac{\kappa^2}{2} (\rho_\phi + p_\phi + \rho_m + 4\rho_r / 3).$$

The effective energy and pressure density are

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 - 3\xi H^2 \phi^2,$$

$$p_\phi = \frac{1}{2} \dot{\phi}^2 + 3\xi H^2 \phi^2 + 2\xi \frac{d}{dt} (H \phi^2).$$
Tracker Behavior
Analytic Solutions in RD and MD eras

- $H = \alpha/t$, i.e. $a(t) \propto t^\alpha$, with $\alpha$ constant:

$$\phi(t) = C_1 t^{l_1} + C_2 t^{l_2},$$

where $C_{1,2}$ are constants and

$$l_{1,2} = \frac{1}{2} \left[ \pm \sqrt{(3\alpha - 1)^2 - 24\xi\alpha^2} - (3\alpha - 1) \right].$$

- For $\xi < 0$, the power-index $l_1$ is positive and $l_2$ is negative, corresponding to increasing and decreasing modes, respectively.

- Considering only the increasing mode, i.e., $\phi(t) = C_1 t^{l_1}$. 
We can solve the analytic solution in RD and MD eras:

\[ w_\phi = \frac{1}{3} \left( 2 - \sqrt{1 - 24\xi} \right), \quad \frac{1}{2} \left( 1 - \sqrt{1 - 32\xi/3} \right), \]
\[ \rho_\phi \propto a^{-5+\sqrt{1-24\xi}}, \quad a^{(-9+\sqrt{9-96\xi})/2}, \]

for the RD \((\alpha = 1/2)\) and MD \((\alpha = 2/3)\) eras, respectively.
Tracker Behavior
Analytic Solutions in SD era

- Setting $\rho_r = 0$:
  \[
  \frac{d}{dt} \left[ F(\phi) a^3 H \right] = \frac{\kappa^2}{2} \rho_m a^3 = \frac{3}{2} H_0^2 \Omega_m, \\
  F(\phi) \equiv 1 + \kappa^2 \xi \phi^2.
  \]

- Setting $\rho_m = 0$ and combining with field equation, we can solve the analytic solution:
  \[
  \phi(a) = \pm \frac{\sin \theta}{\sqrt{-\kappa^2 \xi}}, \\
  w_\phi(a) = -1 - \sqrt{-32 \xi/3 \tan \theta}, \\
  \theta(a) \equiv \sqrt{-6 \xi} \ln a + C_3,
  \]
  where $C_3$ is the integration constant.

- There exists one kind of finite-time future singularities, called “sudden singularity”.
The tracker behavior in RD (MD) eras, and the singularity happens in SD era.

The present ($z = 0$) values of ($\Omega_m, w_\phi$) with $\xi = -0.35$. 

\[ \Omega_m, w_\phi \]
Combining SNIa, BAO and CMB data:
Summary

- Analytic solution exist in this potential-free case.
- Equation of state \( (w_\phi) \) has a tracker behavior and only depend on \( \xi \) in RD and MD eras.
- The final energy density depends on \( \xi \) and initial condition parameter \( C_1 \).
- The observational data can be fitted well but the concordance region for all data is only at the 3\( \sigma \) level.